Cooper-pair box

single electron box: $E_{el} = E_c (n - n_g)^2$

$$E_c = \frac{e^2}{2C_\Sigma}$$

$$n_g = \frac{C_g V_g}{e}$$

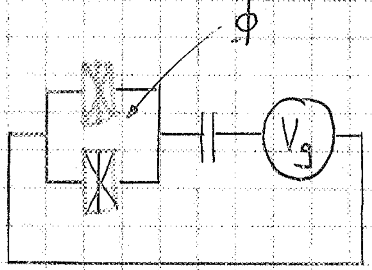
Cooper-pair box $E_{el} = 4 E_c (n - n_g)^2$

$$E_c = \frac{e^2}{2C_\Sigma}$$

$$n_g = \frac{C_g V_g}{2e}$$

Hamilton function

$$\mathcal{H} = 4 E_c (n - n_g)^2 - E_J \cos(\varphi)$$



$E_J = E_J^{max} |\cos(\pi \phi / \phi_0)|$ $\quad \phi = \frac{h}{2e}$

quantizing the circuit

$$\tilde{\mathcal{H}}_{el} = 4 E_c \sum_n (n - n_g)^2 |n\rangle\langle n|$$

charge base $|n\rangle$

$n = \dots -1, 0, 1, \dots$

$$\tilde{n} |n\rangle = n |n\rangle$$

$$\tilde{H}_J = -E_J \cos(\tilde{\varphi})$$

commutation relation $[\tilde{\varphi}, \tilde{n}] = i$

Phase base $|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_n e^{in\varphi} |n\rangle$

$$\cos(\tilde{\varphi}) = \frac{1}{2}\left(e^{i\tilde{\varphi}} + e^{-i\tilde{\varphi}}\right)$$

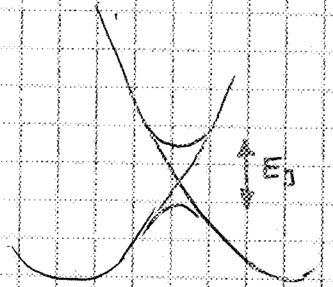
$$e^{i\tilde{\varphi}} |n\rangle = |n+1\rangle$$
$$e^{-i\tilde{\varphi}} |n\rangle = |n-1\rangle$$

$$\tilde{H}_J = -\frac{E_J}{2}\left(\sum_n |n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$

$$\tilde{H} = \tilde{H}_{el} + \tilde{H}_J \qquad \tilde{H}|\psi_n\rangle = E_n |\psi_n\rangle$$

$$\tilde{H} = \begin{pmatrix} \ddots & & & & \\ & 4E_c(-1-n_g)^2 & -E_J/2 & & \\ & -E_J/2 & 4E_c n_g^2 & -E_J/2 & \\ & & -E_J/2 & 4E_c(1-n_g)^2 & -E_J/2 \\ & & & -E_J/2 & 4E_c(2-n_g)^2 & \ddots \end{pmatrix}$$

nearly free electron approximation $E_J \ll E_c$



phase base

$$\tilde{n} = -i \frac{\partial}{\partial \varphi}$$

$$\tilde{H} = 4E_c \left(-i \frac{\partial}{\partial \varphi} - n_g\right)^2 - E_J \cos(\varphi)$$

second order differential equation
Mathieu differential equation

periodic potential ⇒ Bloch bands
quasi momentum corresponds to gate charge

Cooper-pair box qubit

restrict to $0 < n_g < 1$

E_J not too large

zero energy $E_0 = 2E_c(1-2n_g)^2$

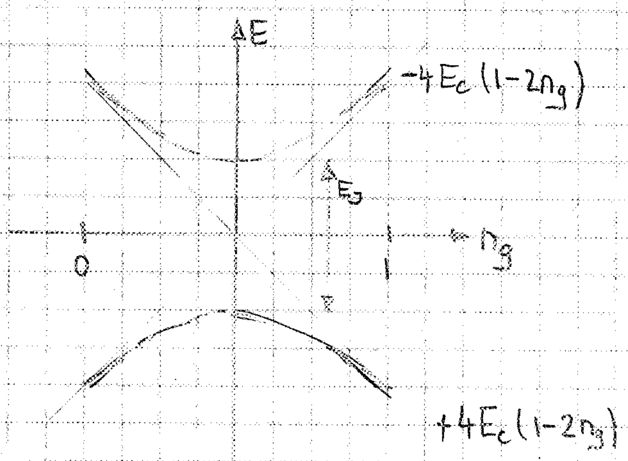
$E_{ch} := 4E_c(1-2n_g)$

restrict to $|0\rangle, |1\rangle$

$$H = -\frac{1}{2}\begin{pmatrix} E_{ch} & E_J \\ E_J & -E_{ch} \end{pmatrix} = -\frac{1}{2}E_{ch}\,\sigma_z - \frac{1}{2}E_J\,\sigma_x$$

$$E_\pm = \pm\frac{1}{2}\sqrt{E_{ch}^2 + E_J^2} = \pm\frac{1}{2}\sqrt{16E_c^2(1-2n_g)^2 + E_J^2}$$

$$\Delta E = E_+ - E_- = \sqrt{16E_c^2(1-2n_g)^2 + E_J^2}$$

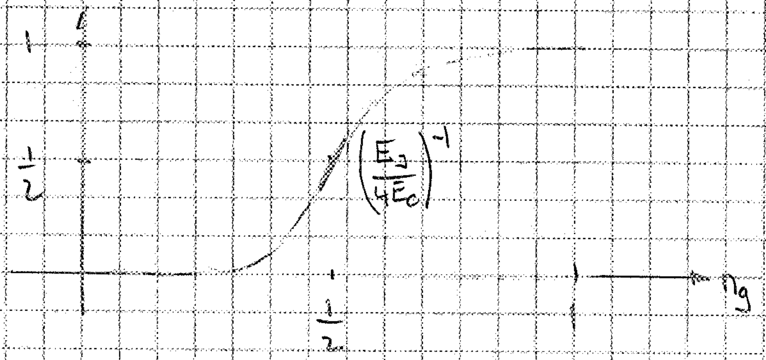


$$|\psi_+\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle - \sin\left(\frac{\theta}{2}\right)|0\rangle$$

$$|\psi_-\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle + \sin\left(\frac{\theta}{2}\right)|0\rangle$$

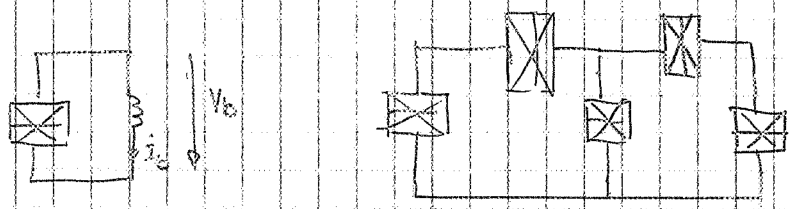
mixing angle $\tan(\theta) = \frac{E_{ch}}{E_J}$

$$\langle n \rangle_{\text{ground state}} = \frac{1}{2}\left(1 - \frac{1-2n_g}{\sqrt{\left(\frac{E_J}{4E_c}\right)^2 + (1-2n_g)^2}}\right)$$



quantizing superconducting circuits

circuit made up of capacitors, inductors, Josephson junctions



branch currents i_b
branch voltage V_b

Kirchhoff's laws

$$\sum_{node} i_b = 0$$

$$\oint_{loop} V_b = 0$$

procedure to find Hamiltonian

write Lagrange function

$$\mathcal{L}(q_1, \ldots q_n, \dot{q}_1, \ldots \dot{q}_n) = T - V$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

- conjugate momentum $p_i(q_i, \dot{q}_i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

- invert the equations to find $\dot{q}_i$

- find Hamilton function with a Legendre transformation

$$\mathcal{H}(q_i, p_i, t) = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$$

$$= \sum_i \dot{q}_i p_i - \mathcal{L}$$

Hamilton's equations

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

- replace variables by operators

$$q_i \rightarrow \hat{q}_i$$
$$p_i \rightarrow \hat{p}_i \qquad [\hat{q}_i, \hat{p}_i] = i\hbar$$
$$\mathcal{H} \rightarrow \hat{\mathcal{H}} \quad \text{Hamiltonian}$$

Note: Circuit has not unique Hamitonian depends on representation

recipe to directly find Hamiltonian

- chose one node as ground
- define a loop-free spanning tree
  each node is linked to ground
  by one and only one path
- define node voltage v_n as
  the sum of branch voltages v_b to ground
- define node current i_n as the
  current flowing to the node by the
  capacitors only

dynamical variables

node flux $\phi_n = \int_{-\infty}^{t} v_n(\tau)\, d\tau$

node charge $q_n = \int_{-\infty}^{t} i(\tau)\, d\tau$

Using Kirchhoff's laws, express branch
flux/charge as as linear combination of
node variables

- sum energies of all branches to get
  Hamilton function

$\phi_n \longrightarrow \hat{\phi}_n$

$q_n \longrightarrow \hat{q}_n \qquad [\hat{\phi}_n, \hat{q}_m] = i\hbar\, \delta_{nm}$